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NONLINEAR SCHROEDINGER EVOLUTION EQUATIONS.(U)

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H. Brezis[1] and T. Gallouet

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ABSTRACT

We consider the nonlinear Schrödinger equation

$$\begin{cases} i \dfrac{\partial u}{\partial t} - \Delta u + k|u|^2 u = 0 & \text{in} \quad \Omega \times [0,\infty) \\ u(x,t) = 0 & \text{in} \quad \partial\Omega \times [0,\infty) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega \end{cases}$$

where Ω is a bounded domain or an exterior domain of $\mathbf{R}^2$. Such an equation has been extensively studied when $\Omega = \mathbf{R}^2$, but the methods do not apply if $\Omega \neq \mathbf{R}^2$. We prove that there exists a unique global smooth solution if $k \geq 0$ or if $k < 0$ and $|k| \int |u_0|^2 < 4$. The proof relies on a new interpolation-embedding inequality:

$$\|u\|_{L^\infty} \leq C[1 + \sqrt{\log(1 + \|u\|_{H^2})}] \quad \text{for every} \quad u \in H^2 \quad \text{with} \quad \|u\|_{H^1} \leq 1 .$$

AMS (MOS) Subject Classifications: 47H15, 46E35, 35K55, 35L60, 45E30

Key Words: Nonlinear Schrödinger equation, Global solutions, Sobolev embedding, Interpolation inequality

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

The nonlinear Schrödinger equation occurs in the study of some problems in nonlinear optics (propagation of laser beams through the atmosphere or in a plasma). ~~It has been considered by many authors.~~ The main novelty of the present result is that it applies to the propagation of beams in channels (instead of the whole space). The proof relies on a new Sobolev-Orlicz embedding inequality which could be useful in other situations.

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NONLINEAR SCHRÖDINGER EVOLUTION EQUATIONS

H. Brezis[1] and T. Gallouet

Let Ω be a domain in $\mathbf{R}^2$ with compact smooth boundary Γ (Ω could be for example a bounded domain or an exterior domain). Consider the equation

$$(1)\qquad \begin{cases} i\dfrac{\partial u}{\partial t} - \Delta u + k|u|^2 u = 0 & \text{in} \quad \Omega \times [0,\infty) \\ u(x,t) = 0 & \text{in} \quad \Gamma \times [0,\infty) \\ u(x,0) = u_0(x) , & \end{cases}$$

where $u(x,t)$ is a complex valued function and $k \in \mathbf{R}$ is a constant. Problem (1) which occurs in nonlinear optics when $\Omega = \mathbf{R}^2$ has been extensively studied in this case (see [1],[2],[3],[5],[8]), but we are not aware of any known result when $\Omega \neq \mathbf{R}^2$.

Our main result is the following:

<u>Theorem 1</u>. Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Assume that <u>one</u> of the following conditions holds

(a) either $k \geq 0$,

(b) or $k < 0$ and $|k| \int |u_0(x)|^2 dx < 4$.

Then there exists a unique solution of (1) such that

$$u \in C([0,\infty);\ H^2(\Omega)) \cap C^1([0,\infty);\ L^2(\Omega)) .$$

The proof of Theorem 1 relies on several Lemmas. The first Lemma is of interest for its own sake; it is a new interpolation-embedding inequality.

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In what follows we denote by C various constants depending only on Ω.

Lemma 2. We have

$$\|u\|_{L^\infty} \le C(1 + \sqrt{\log(1 + \|u\|_{H^2})}) \tag{2}$$

for every $u \in H^2(\Omega)$ with $\|u\|_{H^1} \le 1$.

Proof. It is well known that an H^2 function on Ω can be extended by an H^2 function on $\mathbb{R}^2$. More precisely one can construct an extension operator P such that:

P is a bounded operator from $H^1(\Omega)$ into $H^1(\mathbb{R}^2)$

P is a bounded operator from $H^2(\Omega)$ into $H^2(\mathbb{R}^2)$

$Pu_{|\Omega} = u$ for every $u \in H^1(\Omega)$.

Let $u \in H^2(\Omega)$ with $\|u\|_{H^1} \le 1$. Let $v = Pu$ and denote by $\hat{v}$ the Fourier transform of v. We clearly have

$$\|(1 + |\xi|)\, \hat{v}\|_{L^2(\mathbb{R}^2)} \le C \tag{3}$$

$$\|(1 + |\xi|^2)\, \hat{v}\|_{L^2(\mathbb{R}^2)} \le C\|u\|_{H^2(\Omega)} \tag{4}$$

$$\|u\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\mathbb{R}^2)} \le C\|\hat{v}\|_{L^1(\mathbb{R}^2)} . \tag{5}$$

For $R > 0$ we write

$$\|\hat{v}\|_{L^1} = \int_{|\xi|<R} |\hat{v}(\xi)|\,d\xi + \int_{|\xi|\ge R} |\hat{v}(\xi)|\,d\xi = \int_{|\xi|<R} (1 + |\xi|)\,|\hat{v}(\xi)|\, \frac{1}{1 + |\xi|}\, d\xi$$

$$+ \int_{|\xi|\ge R} (1 + |\xi|^2)\,|\hat{v}(\xi)|\, \frac{1}{1 + |\xi|^2}\, d\xi$$

$$\le C\left[\int_{|\xi|<R} \frac{1}{(1 + |\xi|)^2}\, d\xi\right]^{1/2} + C\|u\|_{H^2} \left[\int_{|\xi|\ge R} \frac{1}{(1 + |\xi|^2)^2}\, d\xi\right]^{1/2}$$

by Cauchy-Schwarz, (3) and (4). A straightforward computation leads to

$$\|\hat{v}\|_{L^1} \leq C[\log(1 + R)]^{1/2} + C\|u\|_{H^2}(1 + R)^{-1}$$

by every $R \geq 0$. We obtain (2) by choosing $R = \|u\|_{H^2}$.

Lemma 3. We have

(6) $$\||u|^2u\|_{H^2} \leq C\|u\|^2_{L^\infty} \|u\|_{H^2} \quad \text{for every } u \in H^2(\Omega) .$$

Proof of Lemma 3. Let D denote any first order differential operator. For $u \in H^2$ we have

$$|D^2(|u|^2u)| \leq C(|u|^2|D^2u| + |u|\,|Du|^2) ,$$

and so

(7) $$\||u|^2u\|_{H^2} \leq C\|u\|^2_{L^\infty}\|u\|_{H^2} + C\|u\|_{L^\infty}\|u\|^2_{W^{1,4}} .$$

On the other hand an inequality of Gagliardo-Nirenberg (see [6]) implies that

(8) $$\|u\|_{W^{1,4}} \leq C\|u\|^{1/2}_{L^\infty} \|u\|^{1/2}_{H^2} .$$

Combining (7) and (8) we obtain (6).

Finally we recall the following well known result essentially due to Segal [7]:

Lemma 4. Assume H is a Hilbert space and $A : D(A) \subset H \to H$ is an m-accretive linear operator. Assume F is a mapping from $D(A)$ into itself which is Lipschitz on every bounded set of $D(A)$. Then for every $u_0 \in D(A)$, there exists a unique solution u of the equation

$$\begin{cases} \dfrac{du}{dt} + Au = Fu \\ u(0) = u_0 \end{cases}$$

defined for $t \in [0,T_{max})$ such that

$$u \in C^1([0,T_{max}); H) \cap C([0,T_{max}); D(A))$$

with the additional property that

$$\begin{cases} \text{either } T_{max} = \infty \\ \text{or } T_{max} < \infty \text{ and } \lim\limits_{t \uparrow T_{max}} \|u(t)\| + \|Au(t)\| = \infty. \end{cases}$$

Proof of Theorem 1. We apply Lemma 4 in $H = L^2(\Omega)$ to $Au = i\Delta u$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $Fu = ik|u|^2 u$. We shall show that $T_{max} = \infty$ by proving that $\|u(t)\|_{H^2}$ remains bounded on every finite time interval.

First we multiply (1) by $\bar{u}$ and consider the imaginary part. This leads to

(9) $$\|u(t)\|_{L^2} = \|u_0\|_{L^2} .$$

Next we multiply (1) by $\frac{\partial \bar{u}}{\partial t}$ and consider the real part. This leads to

(10) $$\frac{1}{2} \int |\nabla u(x,t)|^2 dx + \frac{k}{4} \int |u(x,t)|^4 dx \equiv E_0$$

where

$$E_0 = \frac{1}{2} \int_\Omega |\nabla u_0(x)|^2 dx + \frac{k}{4} \int_\Omega |u_0(x)|^4 dx .$$

We claim that $\|u(t)\|_{H^1}$ remains bounded for $t > 0$. Indeed, this is clear when $k \geq 0$. While if $k < 0$ we have

(11) $$\int |\nabla u(x,t)|^2 \leq \frac{|k|}{2} \int |u(x,t)|^4 dx + 2 E_0 .$$

On the other hand an inequality of Gagliardo and Nirenberg ([6]) shows that(1)

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(1) In order to obtain the constant $1/2$ one proceeds as follows. For $\varphi \in C_0^\infty(\mathbb{R}^2)$ we have $|\varphi(x_1,x_2)| \leq \frac{1}{2} \int_{-\infty}^{+\infty} |\varphi_{x_1}(t,x_2)|dt$, $|\varphi(x_1,x_2)| \leq \frac{1}{2} \int_{-\infty}^{+\infty} |\varphi_{x_2}(x_1,s)|ds$. Thus $\int_{\mathbb{R}^2} |\varphi|^2 dx \leq \frac{1}{4} \int_{\mathbb{R}^2} |\varphi_{x_1}|dx \int_{\mathbb{R}^2} |\varphi_{x_2}|dx$. Choosing $\varphi = |u|^2$ leads to

$$\int |u|^4 dx \leq \int |u|^2 dx (\int |u_{x_1}|^2 dx)^{1/2} (\int |u_{x_2}|^2 dx)^{1/2} \leq \frac{1}{2} \int |u|^2 dx \int |\nabla u|^2 dx.$$

(12) $$\int |u|^4 dx \le \frac{1}{2} \int |u|^2 dx \int |\nabla u|^2 dx$$
$$= \frac{1}{2} \int |u_0|^2 dx \int |\nabla u|^2 dx \ .$$

Combining (11), (12) and assumption (b) in Theorem 1 we see that

(13) $$\|u(t)\|_{H^1} \le C$$

where C is independent of t.

We now denote by $S(t)$ the L^2 isometry group generated by $-A$. From (1) we have

$$u(t) = S(t)u_0 + ik \int_0^t S(t-s)|u(s)|^2 u(s)ds$$

and so

$$Au(t) = S(t)Au_0 + ik \int_0^t S(t-s)A[|u(s)|^2 u(s)]ds \ .$$

Thus

(14) $$\|Au(t)\|_{L^2} \le \|Au_0\|_{L^2} + |k| \int_0^t \|A[|u(s)|^2 u(s)]\|_{L^2} ds \ .$$

Lemma 3 implies that

$$\|A[|u(s)|^2 u(s)]\|_{L^2} \le C\|u(s)\|^2_{L^\infty} \|u(s)\|_{H^2} \ .$$

From Lemma 2 and estimate (13) we deduce that

$$\|u(s)\|_{L^\infty} \le C(1 + \sqrt{\log(1 + \|u(s)\|_{H^2})}) \ .$$

Hence (14) leads to

(15) $$\|u(t)\|_{H^2} \le C + C \int_0^t \|u(s)\|_{H^2}[1 + \log(1 + \|u(s)\|_{H^2})]ds \ .$$

We denote by $G(t)$ the RHS in (15); thus

$$G'(t) = C\|u(t)\|_{H^2}[1 + \log(1 + \|u(t)\|_{H^2})] \le CG(t)[1 + \log(1 + G(t))] .$$

Consequently

$$\frac{d}{dt} \log[1 + \log(1 + G(t))] \le C$$

and we find an estimate for $\|u(t)\|_{H^2}$ of the form

$$\|u(t)\|_{H^2} \le e^{\alpha e^{\beta t}}$$

for some constants α and β. Therefore $\|u(t)\|_{H^2}$ remains bounded on every finite time interval and so we must have $T_{max} = \infty$.

Remarks. 1) The proof of Theorem 1 leads to an estimate of the form $\|u(t)\|_{L^\infty} \le \alpha e^{\beta t}$. We do not know whether $\|u(t)\|_{L^\infty}$ remains actually bounded as $t \to \infty$.

2) When $k < 0$ and $|k| \int |u_0|^2 > 4$, it is known (see [4],[2]) if $\Omega = \mathbf{R}^2$ that the solution of (1) corresponding to some initial conditions may blow up in finite time. A similar phenomenon presumably occurs when $\Omega \neq \mathbf{R}^2$.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We consider the nonlinear Schrödinger equation

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20. ABSTRACT - Cont'd.

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